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A THEORY OF ALGEBRAIC INTEGRATION

R. CASALBUONI

Department of Physics, University of Florence,
Via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy

In this paper we extend the idea of integration to generic algebras. In particular we concentrate over a class of algebras, that we will call self-conjugated, having the property of possessing equivalent right and left multiplication algebras. In this case it is always possible to define an integral sharing many of the properties of the usual integral. For instance, if the algebra has a continuous group of automorphisms, the corresponding derivations are such that the usual formula of integration by parts holds. We discuss also how to integrate over subalgebras. Many examples are discussed, starting with Grassmann algebras, where we recover the usual Berezin's rule. The paraGrassmann algebras are also considered, as well as the algebra of matrices. Since Grassmann and paraGrassmann algebras can be represented by matrices we show also that their integrals can be seen in terms of traces over the corresponding matrices. An interesting application is to the case of group algebras where we show that our definition of integral is equivalent to a sum over the unitary irreducible representations of the group. We show also some example of integration over non self-conjugated algebras (the bosonic and the q-bosonic oscillators), and over non-associative algebras (the octonions).

Contents

1 Introduction

1.1 Motivations

The very idea of supersymmetry leads to the possibility of extending ordinary classical mechanics to more general cases in which ordinary configuration variables live together with Grassmann variables. More recently the idea of extending classical mechanics to more general situations has been further emphasized with the introduction of quantum groups, non-commutative geometry, etc. In order to quantize these general theories, one can try two ways: i) the canonical formalism, ii) the path-integral quantization. In refs.[1,2] classical theories involving Grassmann variables were quantized by using the canonical formalism. But in this case, also the second possibility can be easily realized by using the Berezin's rule for integrating over a Grassmann algebra.[3] It would be desirable to have a way to perform the quantization of theories defined in a general algebraic setting. In this paper we will make a first step toward this construction, that is we will give general rules allowing the possibility of integrating over a given algebra. Given these rules, the next step would be the definition of the path-integral. In order to define the integration rules we will need some guiding principle. So let us start by reviewing how the integration over Grassmann variables comes about. The standard argument for the Berezin's rule is translational invariance. In fact, this guarantees the validity of the quantum action principle. However, this requirement seems to be too technical and we would rather prefer to rely on some more physical argument, as the one which is automatically satisfied by the path integral representation of an amplitude, that is the combination law for probability amplitudes. This is a simple consequence of the factorization properties of the functional measure and of the additivity of the action. In turn, these properties follow in a direct way from the very construction of the path integral starting from the ordinary quantum mechanics. We recall that the construction consists in the computation of the matrix element $\langle q_f, t_f|q_i, t_i\rangle$, $(t_i < t_f)$ by inserting the completeness relation

$$\int dq\, |q, t\rangle\langle q, t| = 1 \tag{1}$$

inside the matrix element at the intermediate times t_a $(t_i < t_a < t_f,\ a = 1, \cdots, N)$, and taking the limit $N \to \infty$ (for sake of simplicity we consider here the quantum mechanical case of a single degree of freedom). The relevant information leading to the composition law is nothing but the completeness relation (1). Therefore we will assume the completeness as the basic principle to use in order to define the integration rules over a generic algebra. In this paper we will limit our task to the construction of the integration rules, and we will not do any attempt to construct the functional integral in the general case. The extension of the relation (1) to a configuration space different from the usual one is far from being trivial. However, we can use an approach that has been largely used in the study of non-commutative geometry [4] and of quantum groups.[5] The approach starts from the observation that in the normal case one can reconstruct a space from the algebra of its functions . Giving this fact, one lifts all the necessary properties in the function space and avoids to work on the space itself. In this way can deal with cases in which no concrete realization of the space itself exists. We will show how to extend the relation (1) to the algebra of functions. In Section 2 we will generalize these considerations to the case of an arbitrary algebra. In Section 3 we will discuss numerous examples of our procedure. Other examples will be given in Sections 4 and 5. The approach to the integration of Grassmann algebras starting from the requirement of completeness, which inspired the present work, was discussed long ago by Martin.[6]

1.2 The algebra of functions

Let us consider a quantum dynamical system and an operator having a complete set of eigenfunctions. For instance one can consider a one-dimensional free particle. The hamiltonian eigenfunctions are

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} \exp{(-ikx)} \tag{2}$$

Or we can consider the orbital angular momentum, in which case the eigenfunctions are the spherical harmonics $Y_\ell^m(\Omega)$. In general the eigenfunctions satisfy orthogonality relations

$$\int \psi_n^*(x)\psi_m(x)\; dx = \delta_{nm} \tag{3}$$

(we will not distinguish here between discrete and continuum spectrum). However $\psi_n(x)$ is nothing but the representative in the $\langle x|$ basis of the eigenkets $|n\rangle$ of the hamiltonian

$$\psi_n(x) = \langle x|n\rangle \tag{4}$$

Therefore the eq. (3) reads

$$\int \langle n|x\rangle\langle x|m\rangle\; dx = \delta_{nm} \tag{5}$$

which is equivalent to say that the $|x\rangle$ states form a complete set and that $|n\rangle$ and $|m\rangle$ are orthogonal. But this means that we can implement the completeness in the $|x\rangle$ space by using the orthogonality relation obeyed by the eigenfunctions defined over this space. On the other side, given this equation, and the completeness relation for the set $\{|\psi_n\rangle\}$, we can reconstruct the completeness in the original space $\mathbb{R}^1$, that is the integration over the line. Now, we can translate the completeness of the set $\{|\psi_n\rangle\}$, in the following two statements

1. The set of functions $\{\psi_n(x)\}$ span a vector space.
2. The product $\psi_n(x)\psi_m(x)$ can be expressed as a linear combination of the functions $\psi_n(x)$, since the set $\{\psi_n(x)\}$ is complete.

All this amounts to say that the set $\{\psi_n(x)\}$ is a basis of an algebra. The product rules for the eigenfunctions are

$$\psi_m(x)\psi_n(x) = \sum_p c_{nmp}\psi_p(x) \tag{6}$$

with

$$c_{nmp} = \int \psi_n(x)\psi_m(x)\psi_p^*(x)\, dx \tag{7}$$

For instance, in the case of the free particle

$$c_{kk'k''} = \frac{1}{\sqrt{2\pi}}\delta(k + k' - k'') \tag{8}$$

Analogously, for the angular momentum, one has the product formula [7]

$$\begin{aligned} Y_{\ell_1}^{m_1}(\Omega)Y_{\ell_2}^{m_2}(\Omega) &= \sum_{L=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{M=-L}^{+L} \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}\right] \\ &\times \langle \ell_1\ell_2 00|L0\rangle\langle \ell_1\ell_2 m_1 m_2|LM\rangle Y_L^M(\Omega) \end{aligned} \tag{9}$$

where $\langle j_1 j_1 m_1 m_2|JM\rangle$ are the Clebsch-Gordan coefficients. A set of eigenfunctions can then be considered as a basis of the algebra (6), with structure constants given by (7). Any function can be expanded in terms of the complete set $\{\psi_n(x)\}$, and therefore it will be convenient, for the future, to introduce the following ket made up in terms of elements of the set $\{\psi_n(x)\}$

$$|\psi\rangle = \begin{pmatrix} \psi_0(x) \\ \psi_1(x) \\ \cdots \\ \psi_n(x) \\ \cdots \end{pmatrix} \tag{10}$$

A function $f(x)$ such that

$$f(x) = \sum_n a_n \psi_n(x) \tag{11}$$

can be represented as

$$f(x) = \langle a|\psi\rangle \tag{12}$$

where

$$\langle a| = (a_0, a_1, \cdots, a_n, \cdots) \tag{13}$$

To write the orthogonality relation in terms of this new formalism it is convenient to realize the complex conjugation as a linear operation on the set $\{\psi_n(x)\}$. In fact, due to the completeness, $\psi_n^*(x)$ itself can be expanded in terms of $\psi_n(x)$

$$\psi_n^*(x) = \sum_n C_{nm}\psi_m(x), \quad C_{nm} = \int dx\, \psi_n^*(x)\psi_m^*(x) \tag{14}$$

or

$$|\psi^*\rangle = C|\psi\rangle \tag{15}$$

Defining a bra as the transposed of the ket $|\psi\rangle$

$$\langle\psi| = (\psi_0(x), \psi_1(x), \cdots (x), \psi_n(x), \cdots) \tag{16}$$

the orthogonality relation becomes

$$\int |\psi^*\rangle\langle\psi|\; dx = \int C|\psi\rangle\langle\psi|\; dx = 1 \tag{17}$$

Notice that by taking the complex conjugate of eq. (15), we get

$$C^* C = 1 \tag{18}$$

The relation (17) makes reference only to the elements of the algebra of functions and it is the key element in order to define the integration rules on the algebra. In fact, we can now use the algebra product to reduce the expression (17) to a linear form

$$\delta_{nm} = \sum_\ell \int \psi_n(x)\psi_\ell(x)C_{\ell m}\; dx = \sum_{\ell,p} c_{n\ell p}C_{\ell m} \int \psi_p(x)\; dx \tag{19}$$

If the set of equations

$$\sum_p A_{nmp} \int \psi_p(x)\; dx = \delta_{nm}, \quad A_{nmp} = \sum_\ell c_{n\ell p}C_{\ell m} \tag{20}$$

has a solution for $\int \psi_p(x)\ dx$, then we will be able to define the integration over all the algebra, by linearity. We will show in the following that indeed a solution exists for many interesting cases. For instance a solution always exists, if the constant function is in the set $\{\psi_p(x)\}$. Let us show what we get for the free particle. The matrix C is easily obtained by noticing that

$$\begin{aligned} \left(\frac{1}{\sqrt{2\pi}}\exp(-ikx)\right)^* &= \frac{1}{\sqrt{2\pi}}\exp(ikx) \\ &= \int dk'\delta(k+k')\frac{1}{\sqrt{2\pi}}\exp(-ik'x) \end{aligned} \tag{21}$$

and therefore

$$C_{kk'} = \delta(k+k') \tag{22}$$

It follows

$$A_{kk'k''} = \int dq\ \delta(k'+q)\frac{1}{\sqrt{2\pi}}\delta(q+k-k'') = \frac{1}{\sqrt{2\pi}}\delta(k-k'-k'') \tag{23}$$

from which

$$\delta(k-k') = \int dk'' \int A_{kk'k''}\psi_{k''}(x)dx = \int \frac{1}{2\pi}\exp(-i(k-k')x)dx \tag{24}$$

This example is almost trivial, but it shows how, given the structure constants of the algebra, the property of the exponential of being the Fourier transform of the delta-function follows automatically from the formalism. In fact, what we have really done it has been **to define the integration rules in the x space** by using only the algebraic properties of the exponential. As a result, our integration rules require that the integral of an exponential is a delta-function. One can perform similar steps in the case of the spherical harmonics, where the C matrix is given by

$$C_{(\ell,m),(\ell',m')} = (-1)^m\delta_{\ell,\ell'}\delta_{m,-m'} \tag{25}$$

and then using the constant function $Y_0^0 = 1/\sqrt{4\pi}$, in the completeness relation.

The procedure we have outlined here is the one that we will generalize in the next Section to arbitrary algebras. Before doing that we will consider the possibility of a further generalization. In the usual path-integral formalism sometimes one makes use of the coherent states instead of the position operator eigenstates. In this case the basis in which one considers the wave functions is a basis of eigenfunctions of a non-hermitian operator

$$\psi(z) = \langle\psi|z\rangle \tag{26}$$

with

$$a|z\rangle = |z\rangle z \tag{27}$$

The wave functions of this type close an algebra, as $\langle z^*|\psi^*\rangle$ do. But this time the two types of eigenfunctions are not connected by any linear operation. In fact, the completeness relation is defined on the direct product of the two algebras

$$\int \frac{dz^* dz}{2\pi i} \exp(-z^* z)|z\rangle\langle z^*| = 1 \tag{28}$$

Therefore, in similar situations, we will not define the integration over the original algebra, but rather on the algebra obtained by the tensor product of the algebra times a copy. The copy corresponds to the complex conjugated functions of the previous example.

2 Algebras

2.1 Self-conjugated algebras

We recall here some of the concepts introduced in,[8] in order to define the integration rules over a generic algebra. We start by considering an algebra $\mathcal{A}$ given by $n+1$ basis elements x_i, with $i = 0, 1, \cdots n$ (we do not exclude the possibility of $n \to \infty$, or of a continuous index). We assume the multiplication rules

$$x_i x_j = f_{ijk} x_k \tag{29}$$

with the usual convention of sum over the repeated indices. For the future manipulations it is convenient to organize the basis elements x_i of the algebra in a bra

$$\langle x| = (\, x_0, \quad x_1, \quad \cdots \quad x_n \,) \tag{30}$$

or in the corresponding ket. Important tools for the study of a generic algebra are the **right and left multiplication algebras**. We define the associated matrices by

$$R_i|x\rangle = |x\rangle x_i, \qquad \langle x|L_i = x_i\langle x| \tag{31}$$

For a generic element $a = \sum_i a_i x_i$ of the algebra we have $R_a = \sum_i a_i R_i$, and a similar equation for the left multiplication. In the following we will use also

$$L_i^T|x\rangle = x_i|x\rangle \tag{32}$$

The matrix elements of R_i and L_i are obtained from their definition

$$(R_i)_{jk} = f_{jik}, \qquad (L_i)_{jk} = f_{ikj} \tag{33}$$

The algebra is completely characterized by the structure constants. The matrices R_i and L_i are just a convenient way of encoding their properties. For instance, in the case of associative algebras one has

$$x_i(x_jx_k) = (x_ix_j)x_k \tag{34}$$

implying the following relations (equivalent one with the other)

$$R_iR_j = f_{ijk}R_k, \quad L_iL_j = f_{ijk}L_k, \quad [R_i, L_j^T] = 0 \tag{35}$$

The first two say that R_i and L_i are linear representations of the algebra, called the regular representations. The third that the right and left multiplications commute for associative algebras. In this paper we will be interested in algebras with identity, and such that there exists a matrix C, satisfying

$$L_i = CR_iC^{-1} \tag{36}$$

We will call these algebras self-conjugated. In the case of associative algebras, the condition (36) says that the regular representations (see eq. (35)) spanned by L_i and R_i are equivalent. Therefore, the non existence of the matrix C boils down two the possibility that the associative algebra admits inequivalent regular representations. This happens, for instance, in the case of the bosonic algebra.[8] In all the examples we will consider here, the C matrix turns out to be symmetric

$$C^T = C \tag{37}$$

This condition of symmetry can be interpreted in terms of the opposite algebra $\mathcal{A}^D$, defined by

$$x_i^Dx_j^D = f_{jik}x_k^D \tag{38}$$

The left and right multiplication in the dual algebra are related to those in $\mathcal{A}$ by

$$R_i^D = L_i^T, \quad L_i^D = R_i^T \tag{39}$$

Therefore, in the associative case, the matrices L_i^T are a representation of the dual algebra

$$L_i^TL_j^T|x\rangle = x_jx_i|x\rangle = f_{jik}L_k^T|x\rangle \tag{40}$$

We see that the property $C^T = C$ implies that the relation (36) holds also for the right and left multiplication in the opposite algebra

$$L_i^D = CR_i^DC^{-1} \tag{41}$$

In the case of associative algebras, the requirement of existence of an identity is not a strong one, because we can always extend the given algebra to another associative algebra with identity.[9] An extension of this type exists also for many other algebras, but not for all. For instance, in the case of a Lie algebra one cannot add an identity with respect to the Lie product. For self-conjugated algebras, L_i has an eigenket given by

$$L_i|Cx\rangle = CR_i|x\rangle = |Cx\rangle x_i, \qquad |Cx\rangle = C|x\rangle \tag{42}$$

as it follows from (36) and (31). Then, as explained in the Introduction, we define the integration for a self-conjugated algebra by the formula

$$\int_{(x)} |Cx\rangle\langle x| = 1 \tag{43}$$

where 1 is the identity in the space of the linear mappings on the algebra. In components the previous definition means

$$\int_{(x)} C_{ij}x_jx_k = C_{ij}f_{jkp}\int_{(x)} x_p = \delta_{ik} \tag{44}$$

This equation is meaningful only if it is possible to invert it in terms of $\int_{(x)} x_p$. This is indeed the case if $\mathcal{A}$ is an algebra with identity (say $x_0 = I$), [8] because by taking $k = 0$ in eq. (44), we get

$$\int_{(x)} x_j = (C^{-1})_{j0} \tag{45}$$

We see now the reason for requiring the condition (36). In fact it ensures that the value (45) of the integral of an element of the basis of the algebra gives the solution to the equation (44). In fact we have

$$\int_{(x)} C_{ij}x_jx_k = C_{ij}f_{jkp}C^{-1}_{p0} = (CR_kC^{-1})_{i0} = (L_k)_{i0} = f_{k0i} = \delta_{ik} \tag{46}$$

as it follows from $x_kx_0 = x_k$. Notice that if C is symmetric we can write the integration also as

$$\int_{(x)} |x\rangle\langle Cx| = 1 \tag{47}$$

which is the form we would have obtained if we had started with the same assumptions but with the transposed version of eq. (31). We will define an arbitrary function on the algebra by

$$f(x) = \sum_i f_ix_i \equiv \langle x|f\rangle \tag{48}$$

and its conjugated as

$$f^*(x) = \sum_{ij} \bar{f}_i C_{ij} x_j = \langle f|Cx\rangle \tag{49}$$

where

$$|f\rangle = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \qquad \langle f| = (\,\bar{f}_0 \quad \bar{f}_1 \quad \cdots \quad \bar{f}_n\,) \tag{50}$$

and $\bar{f}_i$ is the complex conjugated of the coefficient f_i belonging to the field $\mathbb{C}$. Then a scalar product on the algebra is given by

$$\langle f|g\rangle = \int_{(x)} \langle f|Cx\rangle\langle x|g\rangle = \int_{(x)} f^*(x)g(x) = \sum_i \bar{f}_i g_i \tag{51}$$

2.2 Non self-conjugated algebras

In order to generalize the case of coherent states seen at the end of Section 1.2 we will consider now the case in which the C matrix does not exist. For associative algebras this happens when the left and right multiplications span inequivalent regular representations. In this case, let us take an isomorphic copy of $\mathcal{A}$, say $\mathcal{A}^*$

$$x_i^* x_j^* = f_{ijk} x_k^* \tag{52}$$

and

$$R_i|x^*\rangle = |x^*\rangle x_i^*, \qquad \langle x^*|L_i = x_i^*\langle x_i^*| \tag{53}$$

with $|x^*\rangle_i = x_i^*$. We then define the integration over the direct product $\mathcal{A}\otimes\mathcal{A}^*$ as

$$\int_{(x,x^*)} |x^*\rangle\langle x| = 1 \tag{54}$$

or

$$\int_{(x,x^*)} x_i^* x_j = \delta_{ij} \tag{55}$$

giving rise to the scalar product

$$\langle f|g\rangle = \int_{(x,x^*)} \bar{f}(x^*)g(x) = \sum_i \bar{f}_i g_i \tag{56}$$

2.3 Algebras with involution

In some case, as for the toroidal algebras,[10] the matrix C turns out to define a mapping which is an involution of the algebra. Let us consider the property of the involution on a given algebra $\mathcal{A}$. An involution is a linear mapping $^* : \mathcal{A} \to \mathcal{A}$, such that

$$(x^*)^* = x, \qquad (xy)^* = y^* x^*, \qquad x, y \in \mathcal{A} \tag{57}$$

Furthermore, if the definition field of the algebra is $\mathbb{C}$, the involution acts as the complex-conjugation on the field itself. Given a basis $\{x_i\}$ of the algebra, the involution can be expressed in terms of a matrix C such that

$$x_i^* = x_j C_{ji} \tag{58}$$

The eqs. (57) imply

$$(x_i^*)^* = x_j^* C_{ji}^* = x_k C_{kj} C_{ji}^* \tag{59}$$

from which

$$CC^* = 1 \tag{60}$$

From the product property applied to the equality

$$R_i |x\rangle = |x\rangle x_i \tag{61}$$

we get

$$(R_i|x\rangle)^* = \langle x^*| R_i^\dagger = \langle x| C R_i^\dagger = (|x\rangle x_i)^* = x_i^* \langle x^*| = x_i^* \langle x| C \tag{62}$$

and therefore

$$\langle x| C R_i^\dagger C^{-1} = x_j C_{ji} \langle x| = \langle x| L_j C_{ji} \tag{63}$$

that is

$$C R_i^\dagger C^{-1} = L_j C_{ji} \tag{64}$$

or

$$C R_{x_i}^\dagger C^{-1} = L_{x_i^*} \tag{65}$$

If R_i and L_i are *-representations, that is

$$R_{x_i}^\dagger = R_{x_i^*} = R_{x_j} C_{ji} \tag{66}$$

we obtain

$$C R_{x_i}^\dagger C^{-1} = C R_{x_i^*} C^{-1} = L_{x_i^*} \tag{67}$$

Since the involution is non-singular, we get

$$CR_iC^{-1} = L_i \tag{68}$$

and comparing with the adjoint of eq. (67), we see that C is a unitary matrix which, from eq. (60), implies $C^T = C$. Therefore we have the theorem:

Given an associative algebra with involution, if the right and left multiplications are *-representations, then the algebra is self-conjugated.

In this case our integration is a *state* in the Connes terminology.[4]

If the C matrix is an involution we can write the integration as

$$\int_{(x)} |x\rangle\langle x^*| = \int_{(x)} |x^*\rangle\langle x| = 1 \tag{69}$$

2.4 Derivations

We will discuss now the derivations on algebras with identity. Recall that a derivation is a linear mapping on the algebra satisfying

$$D(ab) = (Da)b + a(Db), \qquad a, b \in \mathcal{A} \tag{70}$$

We define the action of D on the basis elements in terms of its representative matrix, d,

$$Dx_i = d_{ij}x_j \tag{71}$$

If D is a derivation, then

$$S = \exp(\alpha D) \tag{72}$$

is an automorphism of the algebra. In fact, it is easily proved that

$$\exp(\alpha D)(ab) = (\exp(\alpha D)a)(\exp(\alpha D)b) \tag{73}$$

On the contrary, if $S(\alpha)$ is an automorphism depending on the continuous parameter α, then from (73), the following equation defines a derivation

$$D = \lim_{\alpha \to 0} \frac{S(\alpha) - 1}{\alpha} \tag{74}$$

In our formalism the automorphisms play a particular role. In fact, from eq. (73) we get

$$S(\alpha)(|x\rangle x_i) = (S(\alpha)|x\rangle)(S(\alpha)x_i) \tag{75}$$

and therefore

$$R_i(S(\alpha)|x\rangle) = S(\alpha)(R_i|x\rangle) = S(\alpha)(|x\rangle x_i) = (S(\alpha)|x\rangle)(S(\alpha)x_i) \tag{76}$$

meaning that $S(\alpha)|x\rangle$ is an eigenvector of R_i with eigenvalue $S(\alpha)x_i$. This equation shows that the basis $x_i' = S(\alpha)x_i$ satisfies an algebra with the same structure constants as those of the basis x_i. Therefore the matrices R_i and L_i constructed in the two basis, and as a consequence the C matrix, are identical. In other words, our formulation is invariant under automorphisms of the algebra (of course this is not true for a generic change of basis). The previous equation can be rewritten in terms of the matrix $s(\alpha)$ of the automorphism $S(\alpha)$, as

$$R_i\,(s(\alpha)|x\rangle) = (s(\alpha)|x\rangle)\, s_{ij}x_j = s_{ij}s(\alpha)R_j|x\rangle \tag{77}$$

or

$$s(\alpha)^{-1}R_i s(\alpha) = R_{S(\alpha)x} \tag{78}$$

If the algebra has an identity element, I, (say $x_0 = I$), then

$$Dx_0 = 0 \tag{79}$$

and therefore

$$Dx_0 = d_{0i}x_i = 0 \Longrightarrow d_{0i} = 0 \tag{80}$$

We will prove now some properties of the derivations. First of all, from the basic defining equation (70) we get

$$\begin{aligned} R_i d|x\rangle &= R_i D|x\rangle = D(R_i|x\rangle = D(|x\rangle x_i) \\ &= d|x\rangle x_i + |x\rangle Dx_i = dR_i|x\rangle + R_{Dx_i}|x\rangle \end{aligned} \tag{81}$$

or

$$[R_i, d] = R_{Dx_i} \tag{82}$$

which is nothing but the infinitesimal version of eq. (78). From the integration rules for a self-conjugated algebra with identity we get

$$\int_{(x)} Dx_i = d_{ij}\int_{(x)} x_j = d_{ij}(C^{-1})_{j0} \tag{83}$$

Showing that in order that the derivation D satisfies the integration by parts rule for any function, $f(x)$, on the algebra

$$\int_{(x)} D(f(x)) = 0 \tag{84}$$

the necessary and sufficient condition is

$$d_{ij}(C^{-1})_{j0} = 0 \tag{85}$$

implying that the d matrix must be singular and have $(C^{-1})_{j0}$ as a null eigenvector.

Next we show that, if a derivation satisfies the integration by part formula (84), then the matrix of related automorphism $S(\alpha) = \exp(\alpha D)$ obeys the equation

$$Cs(\alpha)C^{-1} = s^{T-1}(\alpha) \tag{86}$$

and it leaves invariant the integration. The converse of this theorem is also true. Let us start assuming that D satisfies eq. (84), then

$$\begin{aligned} 0 &= \int_{(x)} D(C|x\rangle\langle x|) = \int_{(x)} Cd|x\rangle\langle x| + \int_{(x)} C|x\rangle\langle Dx| \\ &= CdC^{-1} + d^T \end{aligned} \tag{87}$$

that is

$$CdC^{-1} = -d^T \tag{88}$$

The previous expression can be exponentiated obtaining

$$C\exp(\alpha d)C^{-1} = \exp(-\alpha d^T) \tag{89}$$

from which the equation (86) follows, for $s(\alpha) = \exp(\alpha d)$. To show the invariance of the integral, let us consider the following identity

$$1 = \int_{(x)} s^{T-1}|Cx\rangle\langle x|s^T = \int_{(x)} Cs|x\rangle\langle x|s^T = \int_{(x)} C|Sx\rangle\langle Sx| = \int_{(x)} C|x'\rangle\langle x'| \tag{90}$$

where $x' = Sx$, and we have used eq. (86). For any automorphism of the algebra we have

$$\int_{(x')} |Cx'\rangle\langle x'| = 1 \tag{91}$$

since the numerical values of the matrices R_i and L_i, and consequently the C matrix, are left invariant. Comparing eqs. (90) and (91) we get

$$\int_{(x')} = \int_{(x)} \tag{92}$$

On the contrary, if the integral is invariant under an automorphism of the algebra, the chain of equalities

$$1 = \int_{(x')} |Cx'\rangle\langle x'| = \int_{(x)} |Cx'\rangle\langle x'| = \int_{(x)} Cs|x\rangle\langle x|s^T = CsC^{-1}s^T \tag{93}$$

implies eq. (86), together with its infinitesimal version eq. (88). From this (see the derivation in (87)), we see that the following relation holds

$$0 = \int_{(x)} D(C_{ij}x_jx_k) \tag{94}$$

and by taking $x_k = I$,

$$\int_{(x)} Dx_i = 0 \tag{95}$$

for any basis element of the algebra. Therefore we have proven the following theorem:

If a derivation D satisfies the integration by part rule, eq. (84), the integration is invariant under the related automorphism $\exp(\alpha D)$. On the contrary, if the integration is invariant under a continuous automorphism, $\exp(\alpha D)$, the related derivation, D, satisfies (84).

This theorem generalizes the classical result about the Lebesgue integral relating the invariance under translations of the measure and the integration by parts formula.

Next we will show that, always in the case of an associative self-conjugated algebra, $\mathcal{A}$, with identity, there exists a set of automorphisms such that the measure of integration is invariant. This is the of the **inner derivations**. In the case of an associative algebra $\mathcal{A}$ with identity the set coincides with the adjoint representation of Lie $\mathcal{A}$ (the Lie algebra generated by $[a,b] = ab - ba$, for $a,\ b \in \mathcal{A}$). That is

$$d = R_a - L_a^T \tag{96}$$

or

$$Dx_i = x_ia - ax_i = -(adja)_{ij}x_j \tag{97}$$

We can now proof the following theorem:

For an associative self-conjugated algebra with identity, such that $C^T = C$, the measure of integration is invariant under the automorphisms generated by the inner derivations, or, equivalently, the inner derivations satisfy the rule of *integration by parts.*

In fact, this follows at once from eq. (88)

$$CdC^{-1} = C(R_a - L_a^T)C^{-1} = L_a - (C^{T^{-1}} L_a C^T)^T = L_a - R_a^T = -d^T \quad (98)$$

2.5 *Integration over a subalgebra*

Let us start with a self-conjugated algebra $\mathcal{A}$ with generators x_i, $i = 0, \cdots, n$. Let us further suppose that $\mathcal{A}$ has a self-conjugated sub-algebra $\mathcal{B}$ with generators y_α, with $\alpha = 0, \cdots, m$, $m < n$. As a vector space the algebra $\mathcal{A}$ can be decomposed as

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{C} \quad (99)$$

The vector space $\mathcal{C}$ is generated by vectors v_a, with $a = 1, \cdots n - m$. Since $\mathcal{B}$ is a subalgebra we have multiplication rules

$$\begin{aligned} y_\alpha y_\beta &= f_{\alpha\beta\gamma} y_\gamma \\ v_a y_\alpha &= f_{a\alpha\beta} y_\beta + f_{a\alpha b} v_b \\ v_a v_b &= f_{abc} v_c + f_{ab\alpha} y_\alpha \end{aligned} \quad (100)$$

By definition the integration is defined both in $\mathcal{A}$ and in $\mathcal{B}$. Our aim is to reconstruct the integration over $\mathcal{B}$ as an integration over $\mathcal{A}$ with a convenient measure. To this end, let us consider the matrix S which realizes the change of basis from x_i to (y_α, v_a), that is

$$y_\alpha = S_{\alpha i} x_i, \qquad v_a = S_{ai} x_i \quad (101)$$

This matrix is invertible by hypothesis, and we can reconstruct the original basis as

$$x_i = (S^{-1})_{i\alpha} y_\alpha + (S^{-1})_{ia} v_a \quad (102)$$

To reconstruct the integration over $\mathcal{B}$ in terms of an integration over $\mathcal{A}$, we will construct a function on the algebra

$$P = p_i x_i \quad (103)$$

such that

$$\int_{(\mathcal{A})} v_a P = 0, \qquad \int_{(\mathcal{A})} y_\alpha P = \int_{(\mathcal{B})} y_\alpha \quad (104)$$

These are equivalent to require

$$\int_{(\mathcal{A})} \mathcal{A} P = \int_{(\mathcal{A})} \mathcal{B} P = \int_{(\mathcal{B})} B \quad (105)$$

These are $n+1$ conditions over the $n+1$ unknown p_i. We will see immediately that there is one and only one solution to the problem. In fact, by using the matrix S we can make more explicit the previous equations by writing

$$0 = \int_{(\mathcal{A})} v_a P = S_{ai} p_j \int_{(\mathcal{A})} x_i x_j \tag{106}$$

Recalling that by definition $\mathcal{A}$ is

$$\int_{(\mathcal{A})} x_i x_j = (C_{\mathcal{A}}^{-1})_{ij} \tag{107}$$

we get

$$0 = S_{ai} p_j (C_{\mathcal{A}}^{-1})_{ij} \tag{108}$$

and in analogous way

$$(SC_{\mathcal{A}}^{-1})_{\alpha j} p_j = \int_{(\mathcal{B})} y_\alpha \tag{109}$$

from which we obtain

$$(SC_{\mathcal{A}}^{-1})_{\alpha j} p_j = (C_{\mathcal{B}}^{-1})_{\alpha 0} \tag{110}$$

Since both S and C are invertible, the problem has a unique solution given by

$$p_i = (C_{\mathcal{A}} S^{-1})_{i\alpha} (C_{\mathcal{B}}^{-1})_{\alpha 0} \tag{111}$$

2.6 Change of variables

Consider again a self-conjugated algebra and the following linear change of variables

$$x_i' = S_{ij} x_j \tag{112}$$

The integration rules with respect to the new variables are

$$\int_{x'} x_i' x_j' = (C'^{-1})_{ij} \tag{113}$$

where C' satisfies

$$L_i' = C' R_i' C'^{-1} \tag{114}$$

and the right and left multiplications in the new basis are related to the ones in the old basis in the following manner. From

$$R_i' |x'\rangle = |x'\rangle x_i' \tag{115}$$

we get

$$R'_i S|x\rangle = S|x\rangle S_{ij} x_j = S_{ij} S R_j |x\rangle \tag{116}$$

or

$$R'_i = S_{ij} S R_j S^{-1} \tag{117}$$

and analogously

$$L'_i = S_{ij} S^{T-1} L_j S^T \tag{118}$$

In the new basis the R- and L-representations are still equivalent ($L'_i = C' R'_i C'^{-1}$) implying

$$S^{T-1} L_i S^T = C' S R_i S^{-1} C'^{-1} \tag{119}$$

or

$$L_i = \left(S^T C' S\right) R_i \left(S^T C' S\right)^{-1} \tag{120}$$

Therefore we must have ($L_i = C R_i C^{-1}$)

$$C = S^T C' S A \tag{121}$$

with A invertible and

$$[R_i, A] = 0 \tag{122}$$

We get

$$(C'^{-1})_{ij} = \int_{x'} x'_i x'_j = \int_{x'} S_{il} S_{jm} x_l x_m \tag{123}$$

from which

$$\int_{x'} x_l x_m = (S^{-1} C'^{-1} S^{T^{-1}})_{lm} \tag{124}$$

and in particular

$$\int_{x'} x_i = (S^{-1} C'^{-1} S^{T^{-1}})_{i0} \tag{125}$$

The result can also be expressed in terms of the matrix A defined in eq. (121)

$$A = S^{-1} C'^{-1} S^{T^{-1}} C \tag{126}$$

obtaining

$$\int_{x'} x_i x_j = (A C^{-1})_{ij} \tag{127}$$

3 Examples of Associative Self-Conjugated Algebras

3.1 The Grassmann algebra

We will discuss now the case of the Grassmann algebra $\mathcal{G}_1$, with generators $1, \theta$, such that $\theta^2 = 0$. The multiplication rules are

$$\theta^i \theta^j = \theta^{i+j}, \quad i, j, i+j = 0, 1 \tag{128}$$

and zero otherwise. From the multiplication rules we get the structure constants

$$f_{ijk} = \delta_{i+j,k}, \qquad i,\ j,\ k = 0, 1 \tag{129}$$

from which the explicit expressions for the matrices R_i and L_i follow

$$\begin{aligned}
(R_0)_{ij} &= f_{i0j} = \delta_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(R_1)_{ij} &= f_{i1j} = \delta_{i+1,j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
(L_0)_{ij} &= f_{0ji} = \delta_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(L_1)_{ij} &= f_{1ji} = \delta_{i,j+1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{130}$$

Notice that R_1 and L_1 are nothing but the ordinary annihilation and creation Fermi operators with respect to the vacuum state $|0\rangle = (1, 0)$. The C matrix exists and it is given by

$$(C)_{ij} = \delta_{i+j,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{131}$$

The eigenket of R_i is

$$|\theta\rangle = \begin{pmatrix} 1 \\ \theta \end{pmatrix} \tag{132}$$

and the completeness reads

$$\int_{\mathcal{G}_1} |\theta\rangle\langle\theta| = \int_{\mathcal{G}_1} \begin{pmatrix} \theta & 0 \\ 1 & \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{133}$$

or

$$\int_{\mathcal{G}_1} \theta^i \theta^{1-j} = \delta_{i,j} \tag{134}$$

which means

$$\int_{\mathcal{G}_1} 1 = 0, \qquad \int_{\mathcal{G}_1} \theta = 1 \tag{135}$$

The case of a Grassmann algebra $\mathcal{G}_n$, which consists of 2^n elements obtained by n anticommuting generators $\theta_1, \theta_2, \cdots, \theta_n$, the identity, 1, and by all their products, can be treated in a very similar way. In fact, this algebra can be obtained by taking a convenient tensor product of n Grassmann algebras $\mathcal{G}_1$, which means that the eigenvectors of the algebra of the left and right multiplications are obtained by tensor product of the eigenvectors of eq. (31). The integration rules extended by the tensor product give

$$\int_{\mathcal{G}_n} \theta_n \theta_{n-1} \cdots \theta_1 = 1 \tag{136}$$

and zero for all the other cases, which is equivalent to require for each copy of $\mathcal{G}_1$ the equations (135). It is worth to mention the case of the Grassmann algebra $\mathcal{G}_2$ because it can be obtained by tensor product of $\mathcal{G}_1$ times a copy $\mathcal{G}_1^*$. Then we can apply our second method of getting the integration rules and show that they lead to the same result with a convenient interpretation of the measure. The algebra $\mathcal{G}_2$ is generated by θ_1, θ_2. An involution of the algebra is given by the mapping

$$^* : \qquad \theta_1 \leftrightarrow \theta_2 \tag{137}$$

with the further rule that by taking the * of a product one has to exchange the order of the factors. It will be convenient to put $\theta_1 = \theta$, $\theta_2 = \theta^*$. This allows us to consider $\mathcal{G}_2$ as $\mathcal{G}_1 \otimes \mathcal{G}_1^* \equiv (\mathcal{G}_1, ^*)$. Then the ket and bra eigenvectors of left and right multiplication in $\mathcal{G}_1$ and $\mathcal{G}_1^*$ respectively are given by

$$\langle\theta| = (1, \theta) \qquad |\theta^*\rangle = \begin{pmatrix} 1 \\ \theta^* \end{pmatrix} \tag{138}$$

with

$$R_i|\theta^*\rangle = |\theta^*\rangle {\theta^*}^i, \qquad \langle\theta|L_i = \theta^i \langle\theta| \tag{139}$$

The completeness relation reads

$$\int_{(\mathcal{G}_1, ^*)} |\theta^*\rangle\langle\theta| = \int_{(\mathcal{G}_1, ^*)} \begin{pmatrix} 1 & \theta \\ \theta^* & \theta^*\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{140}$$

This implies

$$\begin{aligned} \int_{(\mathcal{G}_1, ^*)} 1 &= \int_{(\mathcal{G}_1, ^*)} \theta^*\theta = 1 \\ \int_{(\mathcal{G}_1, ^*)} \theta &= \int_{(\mathcal{G}_1, ^*)} \theta^* = 0 \end{aligned} \tag{141}$$

These relations are equivalent to the integration over $\mathcal{G}_2$ if we do the following identification

$$\int_{(\mathcal{G}_1,^*)} = \int_{\mathcal{G}_2} \exp(\theta^*\theta) \tag{142}$$

The origin of this factor can be traced back to the fact that we have

$$\langle\theta|\theta^*\rangle = 1 + \theta\theta^* = \exp(-\theta^*\theta) \tag{143}$$

3.2 The Paragrassmann algebra

We will discuss now the case of a paragrassmann algebra of order p, $\mathcal{G}_1^p$, with generators 1, and θ, such that $\theta^{p+1} = 0$. The multiplication rules are defined by

$$\theta^i\theta^j = \theta^{i+j}, \quad i, j, i + j = 0, \cdots, p \tag{144}$$

and zero otherwise. From the multiplication rules we get the structure constants

$$f_{ijk} = \delta_{i+j,k}, \quad i,\ j,\ k = 0, 1, \cdots, p \tag{145}$$

from which we obtain the following expressions for the matrices R_i and L_i:

$$(R_i)_{jk} = \delta_{i+j,k}, \quad (L_i)_{jk} = \delta_{i+k,j}, \quad i, j, k = 0, 1 \cdots, p \tag{146}$$

The C matrix exists and it is given by

$$(C)_{ij} = \delta_{i+j,p} \tag{147}$$

In fact

$$(CR_iC^{-1})_{lq} = \delta_{l+m,p}\delta_{i+m,n}\delta_{n+q,p} = \delta_{i+p-l,p-q} = \delta_{i+q,l} = (L_i)_{lq} \tag{148}$$

The ket and the bra eigenvectors of L_i are given by

$$C|\theta\rangle = \begin{pmatrix} \theta^p \\ \theta^{p-1} \\ \cdot \\ 1 \end{pmatrix}, \quad \langle\theta| = (1, \theta \cdots, \theta^p) \tag{149}$$

and the completeness reads

$$\int_{\mathcal{G}_1^p} \theta^{p-i}\theta^j = \delta_{ij} \tag{150}$$

which means

$$\int_{\mathcal{G}_1^p} 1 = \int_{\mathcal{G}_1^p} \theta = \int_{\mathcal{G}_1^p} \theta^{p-1} = 0 \tag{151}$$

$$\int_{\mathcal{G}_1^p} \theta^p = 1 \tag{152}$$

in agreement with the results of ref.[6] (see also [11]).

3.3 The algebra of matrices

Since an associative algebra admits always a matrix representation, it is interesting to consider the definition of the integral over the algebra $\mathcal{A}_N$ of the $N \times N$ matrices. These can be expanded in the following general way

$$A = \sum_{n,m=1}^{N} e^{(nm)} a_{nm} \tag{153}$$

where $e^{(nm)}$ are N^2 matrices defined by

$$e_{ij}^{(nm)} = \delta_i^n \delta_j^m, \qquad i.j = 1, \cdots, N \tag{154}$$

These special matrices satisfy the algebra

$$e^{(nm)} e^{(pq)} = \delta_{mp} e^{(nq)} \tag{155}$$

Therefore the structure constants of the algebra are given by

$$f_{(nm)(pq)(rs)} = \delta_{mp} \delta_{nr} \delta_{qs} \tag{156}$$

from which

$$(R_{(pq)})_{(nm)(rs)} = \delta_{pm} \delta_{qs} \delta_{nr}, \qquad (L_{(pq)})_{(nm)(rs)} = \delta_{qr} \delta_{pn} \delta_{ms} \tag{157}$$

The matrix C can be found by requiring that $|Cx\rangle$ is an eigenstate of L_{pq}, that is

$$(L_{(pq)})_{(nm)(rs)} [F(e)]^{(rs)} = [F(e)]^{(nm)} e^{(pq)} \tag{158}$$

where

$$F(e)^{(nm)} = C_{(nm)(rs)} e^{(rs)} \tag{159}$$

We get

$$[F(e)]^{(qm)} \delta_{pn} = [F(e)]^{(nm)} e^{(pq)} \tag{160}$$

By looking at the eq. (155), we see that this equation is satisfied by

$$[F(e)]^{(rs)} = e^{(sr)} \tag{161}$$

It follows

$$C_{(mn)(rs)} = \delta_{ms}\delta_{nr} \tag{162}$$

It is seen easily that C satisfies

$$C^T = C^* = C, \qquad C^2 = 1 \tag{163}$$

Therefore the matrix algebra is a self-conjugated one. One easily checks that the right multiplications satisfy eq. (67), and therefore C is an involution. More precisely, since

$$e^{(mn)^*} = C_{(mn)(pq)} e^{(pq)} = e^{(nm)} \tag{164}$$

the involution is nothing but the hermitian conjugation

$$A^* = A^\dagger, \quad A \in \mathcal{A}_N \tag{165}$$

The integration rules give

$$(C^{-1})_{(rp)(qs)} = \delta_{rs}\delta_{pq} = \int_{(e)} e^{(rp)} e^{(qs)} = \delta_{pq} \int_{(e)} e^{(rs)} \tag{166}$$

We see that this is satisfied by

$$\int_{(e)} e^{(rs)} = \delta_{rs} \tag{167}$$

This result can be obtained also using directly eq. (45), noticing that the identity of the algebra is given by $I = \sum_n e^{(n,n)}$. Therefore

$$\int_{(e)} e^{(rs)} = \sum_n (C^{-1})_{(rs)(nn)} = \sum_n \delta_{ns}\delta_{nr} = \delta_{rs} \tag{168}$$

and, for a generic matrix

$$\int_{(e)} A = \sum_{m,n=1}^{N} a_{nm} \int_{(e)} e^{(nm)} = Tr(A) \tag{169}$$

Since the algebra of the matrices is associative, the inner derivations are given by

$$D_B A = [A, B] \tag{170}$$

Therefore

$$\int_{(e)} D_B A = \int_{(e)} [A, B] = 0 \tag{171}$$

and we see that the integration by parts formula corresponds to the cyclic property of the trace.

3.4 The subalgebra $\mathcal{A}_{N-1}$

Consider the algebra $\mathcal{A}_N$ of the $N \times N$ matrices, and its subalgebra $\mathcal{A}_{N-1}$. We have the decomposition

$$\mathcal{A}_N = \mathcal{A}_{N-1} \oplus \mathcal{C} \tag{172}$$

with

$$\mathcal{C} = \sum_{i=1}^{N-1} \left(e^{(i,N)} \oplus e^{(N,i)} \right) \oplus e^{(N,N)} \tag{173}$$

and

$$\mathcal{A}_{N-1} = \sum_{i,j=1}^{N-1} \oplus e^{(i,j)} \tag{174}$$

Let us put

$$P = \sum_{i,j=1}^{N} p_{ij} e^{(i,j)} \tag{175}$$

then we require

$$\int_{\mathcal{A}_N} \mathcal{C} P = 0 \tag{176}$$

which implies

$$\int_{\mathcal{A}_N} e^{(i,N)} P = p_{Ni} = 0 \tag{177}$$

and analogously

$$p_{iN} = p_{NN} = 0 \tag{178}$$

The other condition

$$\int_{\mathcal{A}_N} \mathcal{A}_{N-1} P = \int_{\mathcal{A}_{N-1}} \mathcal{A}_{N-1} \tag{179}$$

gives

$$\int_{\mathcal{A}_N} e^{(i,j)} P = p_{ji} = \delta_{ij} \tag{180}$$

Therefore

$$P = \begin{pmatrix} 1_{N-1} & 0 \\ 0 & 0 \end{pmatrix} \tag{181}$$

where 1_{N-1} is the identity matrix in $N-1$ dimensions. This result can be checked immediately by computing the product $A_N P$ with A a generic matrix of $\mathcal{A}_\mathcal{N}$

$$A_N P = \begin{pmatrix} A_{N-1} & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_{N-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{N-1} & 0 \\ C & 0 \end{pmatrix} \tag{182}$$

implying

$$\int_{\mathcal{A}_N} A_N P = Tr(A_N P) = Tr(A_{N-1}) = \int_{\mathcal{A}_{N-1}} A_{N-1} \tag{183}$$

3.5 Paragrassmann algebras as subalgebras of $\mathcal{A}_{p+1}$

A paragrassmann algebra of order p can be seen as a subalgebra of the matrix algebra A_{p+1}. In fact, since this algebra is associative it has a matrix representation (the regular one) in terms the $(p+1)\times(p+1)$ right multiplication matrices, R_i (see eq. (146). These are given by

$$(R_i)_{jk} = \delta_{i+j,k} \tag{184}$$

Defining

$$R_\theta \equiv R_1 \tag{185}$$

we can write, in terms of the matrices defined in eq. (154)

$$R_\theta = \sum_{i=1}^{p} e^{(i,i+1)} \tag{186}$$

and

$$R_\theta^k = \sum_{i=1}^{p+1-k} e^{(i,i+k)} \tag{187}$$

Therefore, the most general function on the paragrassmann algebra (as a subalgebra of the matrices $(p+1)\times(p+1)$) is given by

$$f(R_\theta) = \sum_{i=1}^{p+1} a_i R_\theta^{p+1-i} = \sum_{i=1}^{p+1} a_i \sum_{j=1}^{i} e^{(j,p+1+j-i)} \tag{188}$$

In order to construct the matrix P defined in Section (2.5), let us consider a generic matrix $B \in \mathcal{A}_{p+1}$. We can always decompose it as (see later)

$$B = f(R_\theta) + \tilde{B} \tag{189}$$

In order to construct this decomposition, let us consider the most general $(p+1) \times (p+1)$ matrix. We can write

$$B = \sum_{i,j=1}^{p+1} b_{ij} e^{(ij)} = \sum_{i=1}^{p+1} \sum_{j=1}^{p} b_{ij} e^{(ij)} + \sum_{i=1}^{p+1} b_{i,p+1} e^{(i,p+1)} \tag{190}$$

By adding and subtracting

$$\sum_{i=2}^{p+1} b_{i,p+1} \sum_{j=1}^{i-1} e^{(j,p+1+j-i)} \tag{191}$$

we get the decomposition (189) with

$$f(R_\theta) = \sum_{i=1}^{p+1} b_{i,p+1} R_\theta^{p+1-i} \tag{192}$$

and

$$\tilde{B} = \sum_{i=1}^{p+1} \sum_{j=1}^{p} b_{ij} e^{(ij)} - \sum_{i=2}^{p+1} b_{i,p+1} \sum_{j=1}^{i-1} e^{(j,p+1+j-i)} \tag{193}$$

Now, we can check that the matrix P such that

$$\int_\theta f(\theta) = \int_{A_{p+1}} BP = \int_{A_{p+1}} f(R_\theta) P \tag{194}$$

is given by

$$P = e^{(p+1,1)} \tag{195}$$

In fact, we have

$$\tilde{B} e^{(p+1,1)} = 0 \tag{196}$$

implying

$$B e^{(p+1,1)} = f(R_\theta) e^{(p+1,1)} \tag{197}$$

Furthermore

$$Tr[R_\theta^k e^{(p+1,1)}] = \sum_{i=1}^{p+1-k} Tr[e^{(i,i+k)} e^{(p+1,1)}] = Tr[e^{(p+1-k,1)}] = \delta_{p,k} \tag{198}$$

and therefore

$$\int_\theta \theta^k = Tr[R_\theta^k e^{(p+1,1)}] = \delta_{p,k} \tag{199}$$

showing that $e^{(p+1,1)}$ is the matrix P we were looking for.

We notice that the matrices $\tilde{B}$ and $f(R_\theta)$ appearing in the decomposition (189) can be written more explicitly as

$$\tilde{B} = \begin{pmatrix} \tilde{b}_{1,1} & \tilde{b}_{1,2} & \cdots & \tilde{b}_{1,p} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \tilde{b}_{p,1} & \tilde{b}_{p,2} & \cdots & \tilde{b}_{p,p} & 0 \\ \tilde{b}_{p+1,1} & \tilde{b}_{p+1,2} & \cdots & \tilde{b}_{p+1,p} & 0 \end{pmatrix} \tag{200}$$

and

$$f(R_\theta) = \begin{pmatrix} a_{p+1} & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ 0 & a_{p+1} & a_p & \cdots & a_3 & a_2 \\ 0 & 0 & a_{p+1} & \cdots & a_4 & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_p & a_{p-1} \\ 0 & 0 & 0 & \cdots & a_{p+1} & a_p \\ 0 & 0 & 0 & \cdots & 0 & a_{p+1} \end{pmatrix} \tag{201}$$

The $p \times (p+1)$ parameters appearing in $\tilde{B}$ and the $p+1$ parameters in $f(R_\theta)$ can be easily expressed in terms of the $(p+1) \times (p+1)$ parameters defining the matrix B.

In the particular case of a Grassmann algebra we have

$$R_\theta = e^{(1,2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+, \quad P = e^{(2,1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_- \tag{202}$$

The decomposition in eq. (189), for a 2×2 matrix

$$B = a + b\sigma_3 + c\sigma_+ + d\sigma_- \tag{203}$$

is given by

$$\tilde{B} = b(1+\sigma_3) + d\sigma_-, \qquad f(R_\theta) = f(\sigma_+) = a - b + c\sigma_+ \tag{204}$$

and the integration is

$$\int_{(\theta)} f(\theta) = Tr[f(\sigma_+)\sigma_-] \tag{205}$$

from which

$$\int_{(\theta)} 1 = Tr[\sigma_-] = 0, \qquad \int_{(\theta)} \theta = Tr[\sigma_+\sigma_-] = 1 \tag{206}$$

3.6 Projective group algebras

Let us start defining a projective group algebra. We consider an arbitrary projective linear representation, $a \to x(a)$, $a \in G$, $x(a) \in \mathcal{A}(G)$, of a given group G. The representation $\mathcal{A}(G)$ defines in a natural way an associative algebra with identity (it is closed under multiplication and it defines a generally complex vector space). This algebra will be denoted by $\mathcal{A}(G)$. The elements of the algebra are given by the combinations

$$\sum_{a\in G} f(a)x(a) \tag{207}$$

For a group with an infinite number of elements, there is no a unique definition of such an algebra. The one defined in eq. (207) corresponds to consider a formal linear combination of a finite number of elements of G. This is very convenient since we will not be concerned here with topological problems. Other definitions correspond to take complex functions on G such that

$$\sum_{a\in G} |f(a)| < \infty \tag{208}$$

Or, in the case of compact groups, the sum is defined in terms of the Haar invariant measure. In the following we will not need to be more precise about this point. The basic product rule of the algebra follows from the group property

$$x(a)x(b) = e^{i\alpha(a,b)}x(ab) \tag{209}$$

where $\alpha(a,b)$ is called a cocycle. This is constrained, by the requirement of associativity of the representation, to satisfy

$$\alpha(a,b) + \alpha(ab,c) = \alpha(b,c) + \alpha(a,bc) \tag{210}$$

Changing the element $x(a)$ of the algebra by a phase factor $e^{i\phi(a)}$, that is, defining

$$x'(a) = e^{-i\phi(a)}x(a) \tag{211}$$

we get

$$x'(a)x'(b) = e^{i(\alpha(a,b)-\phi(ab)+\phi(a)+\phi(b))}x'(ab) \tag{212}$$

This is equivalent to change the cocycle to

$$\alpha'(a,b) = \alpha(a,b) - [\phi(ab) - \phi(a) - \phi(b)] \tag{213}$$

In particular, if $\alpha(a,b)$ is of the form $\phi(ab)-\phi(a)-\phi(b)$, it can be transformed to zero, and therefore the corresponding projective representation is isomorphic to a vector one. For this reason the combination

$$\alpha(a,b) = \phi(ab) - \phi(a) - \phi(b) \tag{214}$$

is called a trivial cocycle. Let us now discuss some properties of the cocycles. We start from the relation (e is the identity element of G)

$$x(e)x(e) = e^{i\alpha(e,e)}x(e) \tag{215}$$

By the transformation $x'(e) = e^{-i\alpha(e,e)}x(e)$, we get

$$x'(e)x'(e) = x'(e) \tag{216}$$

Therefore we can assume

$$\alpha(e,e) = 0 \tag{217}$$

Then, from

$$x(e)x(a) = e^{i\alpha(e,a)}x(a) \tag{218}$$

multiplying by $x(e)$ to the left, we get

$$x(e)x(a) = e^{i\alpha(e,a)}x(e)x(a) \tag{219}$$

implying

$$\alpha(e,a) = \alpha(a,e) = 0 \tag{220}$$

where the second relation is obtained in analogous way. Now, taking $c = b^{-1}$ in eq. (210), we get

$$\alpha(a,b) + \alpha(ab,b^{-1}) = \alpha(b,b^{-1}) \tag{221}$$

Again, putting $a = b^{-1}$

$$\alpha(b^{-1},b) = \alpha(b,b^{-1}) \tag{222}$$

We can go farther by considering

$$x(a)x(a^{-1}) = e^{i\alpha(a,a^{-1})}x(e) \tag{223}$$

and defining

$$x'(a) = e^{-i\alpha(a,a^{-1})/2}x(a) \tag{224}$$

from which

$$x'(a)x'(a^{-1}) = e^{-i\alpha(a,a^{-1})}x(a)x(a^{-1}) = x(e) = x'(e) \tag{225}$$

Therefore we can transform $\alpha(a, a^{-1})$ to zero without changing the definition of $x(e)$,

$$\alpha(a, a^{-1}) = 0 \tag{226}$$

As a consequence, equation (221) becomes

$$\alpha(a, b) + \alpha(ab, b^{-1}) = 0 \tag{227}$$

We can get another relation using $x(a^{-1}) = x(a)^{-1}$

$$\begin{aligned} x(a^{-1})x(b^{-1}) &= e^{i\alpha(a^{-1},b^{-1})}x(a^{-1}b^{-1}) = x(a)^{-1}x(b)^{-1} \\ &= (x(b)x(a))^{-1} = e^{-i\alpha(b,a)}x(a^{-1}b^{-1}) \end{aligned} \tag{228}$$

from which

$$\alpha(a^{-1}, b^{-1}) = -\alpha(b, a) \tag{229}$$

and together with eq. (227) we get

$$\alpha(ab, b^{-1}) = \alpha(b^{-1}, a^{-1}) \tag{230}$$

The last two relations will be useful in the following. From the product rule

$$x(a)x(b) = e^{i\alpha(a,b)}x(ab) = \sum_{c\in G} f_{abc}x(c) \tag{231}$$

we get the structure constants of the algebra

$$f_{abc} = \delta_{ab,c}e^{i\alpha(a,b)} \tag{232}$$

The delta function is defined according to the nature of the sum over the group elements.

To define the integration over $\mathcal{A}(G)$, we start as usual by introducing a ket with elements given by $x(a)$, that is $|x\rangle_a = x(a)$, and the corresponding transposed bra $\langle x|$. From the algebra product, we get immediately

$$(R(a))_{bc} = f_{bac} = \delta_{ba,c}e^{i\alpha(b,a)}, \qquad (L(a))_{bc} = f_{acb} = \delta_{ac,b}e^{i\alpha(a,c)} \tag{233}$$

We show now that also these algebras are self-conjugated. Let us look for eigenkets of $L(a)$

$$L(a)|Cx\rangle = |Cx\rangle x(a) \tag{234}$$

giving

$$\delta_{ac,b}(Cx)_c e^{i\alpha(a,c)} = e^{i\alpha(a,a^{-1}b)}(Cx)_{a^{-1}b} = (Cx)_b x(a) \tag{235}$$

By putting

$$(Cx)_a = k_a x(a^{-1}) \tag{236}$$

we obtain

$$k_{a^{-1}b} x(b^{-1}a) e^{i\alpha(a,a^{-1}b)} = k_b e^{i\alpha(b^{-1},a)} x(b^{-1}a) \tag{237}$$

Then, from eqs. (230) and (229)

$$k_{a^{-1}b} = k_b \tag{238}$$

Therefore $k_a = k_e$, and assuming $k_e = 1$, it follows

$$(Cx)_a = x(a^{-1}) = x(a)^{-1} \tag{239}$$

giving

$$C_{a,b} = \delta_{ab,e} \tag{240}$$

This shows also that

$$C^T = C \tag{241}$$

at least in the cases of discrete and compact groups. The mapping $C : \mathcal{A} \to \mathcal{A}$ is an involution of the algebra. In fact, by defining

$$x(a)^* = x(b) C_{b,a} = x(a^{-1}) = x(a)^{-1} \tag{242}$$

we have $(x(a)^*)^* = x(a)$, and $x(b)^* x(a)^* = (x(a)x(b))^*$. From our general rule of integration (see eq. (45)) we get

$$\int_{(x)} x(a) = C^{-1}_{e,a} = \delta_{e,a} \tag{243}$$

Therefore we are allowed to expand a function on the group ($|f\rangle_a = f(a)$) as

$$f(a) = \int_{(x)} x(a^{-1}) \langle x|f\rangle \tag{244}$$

with $\langle x|f\rangle = \sum_{b\in G} x(b) f(b)$. It is also possible to define a scalar product among functions on the group. Defining, $\langle f|_a = \bar{f}(a)$, where $\bar{f}(a)$ is the complex conjugated of $f(a)$, we put

$$\langle f|g\rangle = \int_{(x)} \langle f|Cx\rangle\langle x|g\rangle = \int_{(x)} \bar{f}(x^*) g(x) = \sum_{a\in G} \bar{f}(a) g(a) \tag{245}$$

It is important to stress that this definition depends only on the algebraic properties of $\mathcal{A}(G)$ and not on the specific representation chosen for this construction.

3.7 What is the meaning of the algebraic integration?

As we have said in the previous Section, the integration formula we have obtained is independent on the group representation we started with. In fact, it is based only on the structure of right and left multiplications, that is on the abstract algebraic product. This independence on the representation suggests that in some way we are "summing" over all the representations. To understand this point, we will study in this Section vector representations. To do that, let us introduce a label λ for the vector representation we are actually using to define $\mathcal{A}(G)$. Then a generic function on $\mathcal{A}(G)_\lambda$

$$\hat{f}(\lambda) = \sum_{a \in G} f(a) x_\lambda(a) \tag{246}$$

can be thought as the Fourier transform of the function $f : G \to \mathbb{C}$. Using the algebraic integration we can invert this expression (see eq. (244))

$$f(a) = \int_{(x_\lambda)} \hat{f}(\lambda) x_\lambda(a^{-1}) \tag{247}$$

But it is a well known result of the harmonic analysis over the groups, that in many cases it is possible to invert the Fourier transform, by an appropriate sum over the representations. This is true in particular for finite and compact groups. Therefore the algebraic integration should be the same thing as summing or integrating over the labels λ specifying the representation. In order to show that this is the case, let us recall a few facts about the Fourier transform over the groups.[12] First of all, given the group G, one defines the set $\hat{G}$ of the equivalence classes of the irreducible representations of G. Then, at each point λ in $\hat{G}$ we choose a unitary representation x_λ belonging to the class λ, and define the Fourier transform of the function $f : G \to \mathbb{C}$, by the eq. (246). In the case of compact groups, instead of the sum over the group element one has to integrate over the group by means of the invariant Haar measure. For finite groups, the inversion formula is given by

$$f(a) = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{f}(\lambda) x_\lambda(a^{-1})] \tag{248}$$

where n_G is the order of the group and d_λ the dimension of the representation λ. Therefore, we get the identification

$$\int_{(x)} \{\cdots\} = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\{\cdots\}] \tag{249}$$

A more interesting way of deriving this relation, is to take in (246), $f(a) = \delta_{e,a}$, obtaining for its Fourier transform, $\hat{\delta} = x_\lambda(e) = 1_\lambda$, where the last symbol means the identity in the representation λ. By inserting this result into (248) we get the identity

$$\delta_{e,a} = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{x}_\lambda(a^{-1})] \tag{250}$$

which, compared with eq. (243), gives (249). This shows explicitly that the algebraic integration for vector representations of G is nothing but the sum over the representations of G.

An analogous relation is obtained in the case of compact groups. This can also be obtained by a limiting procedure from finite groups, if we insert $1/n_G$, the volume of the group, in the definition of the Fourier transform. That is one defines

$$\hat{f}(\lambda) = \frac{1}{n_G} \sum_{a \in G} f(a) x_\lambda(a) \tag{251}$$

from which

$$f(a) = \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{f}(\lambda) x_\lambda(a^{-1})] \tag{252}$$

Then one can go to the limit by substituting the sum over the group elements with the Haar measure

$$\hat{f}(\lambda) = \int_G d\mu(a) f(a) x_\lambda(a) \tag{253}$$

The inversion formula (252) remains unchanged. We see that in these cases the algebraic integration sums over the elements of the space $\hat{G}$, and therefore it can be thought as the dual of the sum over the group elements (or the Haar integration for compact groups). By using the Fourier transform (246) and its inversion (247), one can easily establish the Plancherel fromula. In fact by multiplying together two Fourier transforms, one gets

$$\hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{a \in G} \left(\sum_{b \in G} f_1(b) f_2(b^{-1}a) \right) x_\lambda(a) \tag{254}$$

from which

$$\int_{(x)} \hat{f}_1(\lambda)\hat{f}_2(\lambda) x_\lambda(a^{-1}) = \sum_{b \in G} f_1(b) f_2(b^{-1}a) \tag{255}$$

and taking $a = e$ we obtain

$$\int_{(x)} \hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{b \in G} f_1(b) f_2(b^{-1}) \tag{256}$$

This formula can be further specialized, by taking $f_2 \equiv f$ and for f_1 the involuted of f. That is

$$\hat{f}^*(\lambda) = \sum_{a\in G} \bar{f}(a) x_\lambda(a^{-1}) \tag{257}$$

where use has been made of eq. (242). Then, from eq. (256) we get the Plancherel formula

$$\int_{(x)} \hat{f}^*(\lambda)\hat{f}(\lambda) = \sum_{a\in G} \bar{f}(a) f(a) \tag{258}$$

Let us also notice that eq. (254) says that the Fourier transform of the convolution of two functions on the group is the product of the Fourier transforms.

We will consider now projective representations. In this case, the product of two Fourier transforms is given by

$$\hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{a\in G} h(a) x_\lambda(a) \tag{259}$$

with

$$h(a) = \sum_{b\in G} f_1(b) f_2(b^{-1}a) e^{i\alpha(b,b^{-1}a)} \tag{260}$$

Therefore, for projective representations, the convolution product is deformed due to the presence of the phase factor. However, the Plancherel formula still holds. In fact, since in

$$h(e) = \sum_{b\in G} f_1(b) f_2(b^{-1}) \tag{261}$$

using eq. (43), the phase factor disappears and the previous derivation from eq. (256) to eq. (258) is still valid. Notice that eq. (259) tells us that the Fourier transform of the deformed convolution product of two functions on the group, is equal to the product of the Fourier transforms.

3.8 The case of abelian groups

In this Section we consider the case of abelian groups, and we compare the Fourier analysis made in our framework with the more conventional one made in terms of the characters. A fundamental property of the abelian groups is that the set $\hat{G}$ of their vector unitary irreducible representations (VUIR), is itself an abelian group, the dual of G (in the sense of Pontryagin [12]). Since the VUIR's are one-dimensional, they are given by the characters of the group. We will denote the characters of G by $\chi_\lambda(a)$, where $a \in G$, and λ denotes

Table 1: *Parameterization of the abelian group G and of its dual $\hat{G}$, for* $G = \mathbb{R}^D$, Z^D, T^D, Z_n^D.

	$G = R^D$ $\hat{G} = R^D$	$G = Z^D$ $\hat{G} = T^D$	$G = T^D$ $\hat{G} = Z^D$	$G = Z_N^D$ $\hat{G} = Z_N^D$
$\vec{a}$	$-\infty \leq a_i \leq +\infty$	$a_i = \dfrac{2\pi m_i}{L}$ $m_i \in Z$	$0 \leq a_i \leq L$	$a_i = k_i,$ $0 \leq k_i \leq n-1$
$\vec{q}$	$-\infty \leq q_i \leq +\infty$	$0 \leq q_i \leq L$	$q_i = \dfrac{2\pi m_i}{L}$ $m_i \in Z$	$q_i = \dfrac{2\pi \ell_i}{N}$ $0 \leq \ell_i \leq n-1$

the representation of G. For what we said before, the parameters λ can be thought as the elements of the dual group. The parameterization of the group element a and of the representation label λ are given in Table 1, for the most important abelian groups and for their dual groups, where we have used the notation $a = \vec{a}$ and $\lambda = \vec{q}$.

The characters are given by

$$\chi_\lambda(a) \equiv \chi_{\vec{q}}\,(\vec{a}) = e^{-i\vec{q}\cdot\vec{a}} \tag{262}$$

and satisfy the relation (here we use the additive notation for the group operation)

$$\chi_\lambda(a+b) = \chi_\lambda(a)\chi_\lambda(b) \tag{263}$$

and the dual

$$\chi_{\lambda_1+\lambda_2}(a) = \chi_{\lambda_1}(a)\chi_{\lambda_2}(a) \tag{264}$$

That is they define vector representations of the abelian group G and of its dual, $\hat{G}$. Also we can easily check that the operators

$$D_{\vec{q}}\chi_{\vec{q}}\,(\vec{a}) = -i\vec{a}\chi_{\vec{q}}\,(\vec{a}) \tag{265}$$

are derivations on the algebra (263) of the characters for any G in Table 1.

We can use the characters to define the Fourier transform of the function $f(g) : G \to \mathbb{C}$

$$\tilde{f}(\lambda) = \sum_{a\in G} f(a)\chi_\lambda(a) \tag{266}$$

If we evaluate the Fourier transform of the deformed convolution of eq. (260), we get

$$\tilde{h}(\lambda) = \sum_{a\in G} h(a)\chi_\lambda(a) = \sum_{a,b\in G} f(a)\chi_\lambda(a)e^{i\alpha(a,b)}g(b)\chi_\lambda(b) \tag{267}$$

In the case of vector representations the Fourier transform of the convolution is the product of the Fourier transforms. In the case of projective representations, the result, using the derivation introduced before, can be written in terms of the Moyal product (we omit here the vector signs)

$$\tilde{h}(\lambda) = \tilde{f}(\lambda) \star \tilde{g}(\lambda) = e^{-i\alpha(D_{\lambda'},D_{\lambda''})}\tilde{f}(\lambda')\tilde{g}(\lambda'')\Big|_{\lambda'=\lambda''=\lambda} \tag{268}$$

Therefore, the Moyal product arises in a very natural way from the projective group algebra. On the other hand, we have shown in the previous Section, that the use of the Fourier analysis in terms of the projective representations avoids the Moyal product. The projective representations of abelian groups allow a derivation on the algebra, analogous to the one in eq. (265), with very special features. In fact we check easily that

$$\vec{D}x_\lambda(\vec{a}) = -i\vec{a}x_\lambda(\vec{a}) \tag{269}$$

is a derivation, and furthermore

$$\int_{(x_\lambda)} \vec{D}x_\lambda(\vec{a}) = 0 \tag{270}$$

From this it follows, by linearity, that the integral of $\vec{D}$ applied to any function on the algebra is zero

$$\int_{(x_\lambda)} \vec{D}\left(\sum_{a\in G} f(\vec{a})x_\lambda(\vec{a})\right) = 0 \tag{271}$$

This relation is very important because, as we have shown in,[13] the automorphisms generated by $\vec{D}$, that is $\exp(\vec{\alpha}\cdot\vec{D})$, leave invariant the integral. Notice that this derivation generalizes the derivative with respect to the parameter $\vec{q}$, although this has no meaning in the present case. In the case of nonabelian groups, a derivation sharing the previous properties can be defined only if there exists a mapping $\sigma : G \to \mathbb{C}$, such that

$$\sigma(ab) = \sigma(a) + \sigma(b), \qquad a, b \in G \tag{272}$$

since in this case, defining

$$Dx(a) = \sigma(a)x(a) \tag{273}$$

we get

$$\begin{aligned} D(x(a)x(b)) &= \sigma(ab)x(a)x(b) = (\sigma(a) + \sigma(b))x(a)x(b) \\ &= (Dx(a))x(b) + x(a)(Dx(b)) \end{aligned} \tag{274}$$

Having defined derivations and integrals one has all the elements for the harmonic analysis on the projective representations of an abelian group.

Let us start considering $G = R^D$. In the case of vector representations we have

$$x_{\vec{q}}\,(\vec{a}) = e^{-i\vec{q}\cdot\vec{a}} \tag{275}$$

with $\vec{a} \in G$, and $\vec{q} \in \hat{G} = R^D$ labels the representation. The Fourier transform is

$$\hat{f}(\vec{q}) = \int d^D\vec{a} f(\vec{a}) e^{-i\vec{q}\cdot\vec{a}} \tag{276}$$

Here the Haar measure for G coincides with the ordinary Lebesgue measure. Also, since $\hat{G} = R^D$, we can invert the Fourier transform by using the Haar measure on the dual group, that is, again the Lebesgue measure. In the projective case, eq. (275) still holds true, if we assume $\vec{q}$ as a vector operator satisfying the commutation relations

$$[q_i, q_j] = i\eta_{ij} \tag{277}$$

with η_{ij} numbers which can be related to the cocycle, by using the Baker-Campbell-Hausdorff formula

$$e^{-i\vec{q}\cdot\vec{a}} e^{-i\vec{q}\cdot\vec{b}} = e^{-i\eta_{ij}a_i b_j/2} e^{-i\vec{q}\cdot(\vec{a}+\vec{b})} \tag{278}$$

giving

$$\alpha(\vec{a}, \vec{b}) = -\frac{1}{2}\eta_{ij}a_i b_j \tag{279}$$

The inversion of the Fourier transform can now be obtained by our formulation of the algebraic integration in the form

$$f(\vec{a}) = \int_{(\vec{q})} \hat{f}(\vec{q}) x_{\vec{q}}\,(-\vec{a}) \tag{280}$$

where the dependence on the representation is expressed in terms of $\vec{q}$, thought now they are not coordinates on $\hat{G}$. We recall that in this case, eq. (243) gives

$$\int_{(\vec{q})} x_{\vec{q}}\,(\vec{a}) = \delta^D(\vec{a}) \tag{281}$$

Therefore, the relation between the algebraic integration and the Lebesgue integral in $\hat{G}$, in the vector case is

$$\int_{(\vec{q})} = \int \frac{d^D\vec{q}}{(2\pi)^D} \tag{282}$$

In the projective case the right hand side of this relation has no meaning, whereas the left hand side is still well defined. Also, we cannot maintain the interpretation of the q_i's as coordinates on the dual space $\hat{G}$. However, we can define elements of $\mathcal{A}(G)$ having the properties of the q_i's (in particular satisfying eq. (277)), by using the Fourier analysis. That is we define

$$q_i = \int d^D\vec{a}\left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right) x_{\vec{q}}\,(\vec{a}) \tag{283}$$

which is an element of $\mathcal{A}(G)$ obtained by Fourier transforming a distribution over G, which is a honestly defined space. From this definition we can easily evaluate the product

$$q_i x_{\vec{q}}\,(\vec{a}) = \int d^D\vec{b}\left(-i\frac{\partial}{\partial b_i}\delta^D(\vec{b})\right) x_{\vec{q}}\,(\vec{b}) x_{\vec{q}}\,(\vec{a}) \tag{284}$$

Using the algebra and integrating by parts, one gets the result

$$q_i x_{\vec{q}}\,(\vec{a}) = i\nabla_i x_{\vec{q}}\,(\vec{a}) \tag{285}$$

where

$$\nabla_i = \frac{\partial}{\partial a_i} + i\alpha_{ij}a_j \tag{286}$$

where $\alpha_{ij} = \alpha(\vec{e}_{(i)}, \vec{e}_{(j)})$, with $\vec{e}_{(i)}$ an orthonormal basis in $\mathbb{R}^D$. In a completely analogous way one finds

$$x_{\vec{q}}\,(\vec{a})q_i = i\overline{\nabla}_i x_{\vec{q}}\,(\vec{a}) \tag{287}$$

where

$$\overline{\nabla}_i = \frac{\partial}{\partial a_i} - i\alpha_{ij}a_j \tag{288}$$

Then, we evaluate the commutator

$$[q_i, \hat{f}(\vec{q})] = \int d^D\vec{a}\,\left[-i\left(\overline{\nabla}_i - \nabla_i\right) f(\vec{a})\right] x_{\vec{q}}(\vec{a}) \tag{289}$$

where we have done an integration by parts. We get

$$[q_i, \hat{f}(\vec{q})] = -2i\alpha_{ij} D_{q_j} \hat{f}(\vec{q}) \tag{290}$$

where D_{q_j} is the derivation (269), with q_j a reminder for the direction along wich the derivation acts upon. In particular, from

$$D_{q_j} q_i = \int d^D\vec{a} \left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right)(-ia_j)x_{\vec{q}}(\vec{a}) = \delta_{ij} \tag{291}$$

we get

$$[q_i, q_j] = -2i\alpha_{ij} \tag{292}$$

in agreement with eq. (277), after the identification $\alpha_{ij} = -\eta_{ij}/2$.

The automorphisms induced by the derivations (269) are easily evaluated

$$S(\vec{\alpha})x_{\vec{q}}\,(\vec{a}) = e^{\vec{\alpha}\cdot D_{\vec{q}}} x_{\vec{q}}\,(\vec{a}) = e^{-i\vec{\alpha}\cdot\vec{a}} x_{\vec{q}}\,(\vec{a}) = x_{\vec{q}+\vec{\alpha}}(\vec{a}) \tag{293}$$

where the last equality follows from

$$\int d^D\vec{a}\left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right) e^{\vec{\alpha}\cdot D_{\vec{q}}} x_{\vec{q}}\,(\vec{a}) = q_i + \alpha_i \tag{294}$$

Meaning that in the vector case, $S(\vec{\alpha})$ induces translations in $\hat{G}$. Since $D_{\vec{q}}$ satisfies the eq. (271), it follows from Section 2.4 (see also [13]) that the automorphism $S(\vec{\alpha})$ leaves invariant the algebraic integration measure

$$\int_{(\vec{q})} = \int_{(\vec{q}+\vec{\alpha})} \tag{295}$$

This shows that it is possible to construct a calculus completely analogous to the one that we have on $\hat{G}$ in the vector case, just using the Fourier analysis following by the algebraic definition of the integral. We can push this analysis a little bit further by looking at the following expression

$$\int_{(\vec{q})} \hat{f}(\vec{q})\; q_i\; x_{\vec{q}}\,(-\vec{a}) = -i\left(\frac{\partial}{\partial a_i} + i\alpha_{ij}a_j\right) f(\vec{a}) \tag{296}$$

where we have used eq. (285). In the case $D = 2$ this equation has a physical interpretation in terms of a particle of charge e, in a constant magnetic field B. In fact, the commutators among canonical momenta are

$$[\pi_i, \pi_j] = ieB\epsilon_{ij} \tag{297}$$

where ϵ_{ij} is the 2-dimensional Ricci tensor. Therefore, identifying π_i with q_i, we get $\alpha_{ij} = -eB\epsilon_{ij}/2$. The corresponding vector potential is given by

$$A_i(\vec{a}) = -\frac{1}{2}\epsilon_{ij}Ba_j = \frac{1}{e}\alpha_{ij}a_j \tag{298}$$

Then, eq. (296) tells us that the operation $\hat{f}(\vec{q}) \to \hat{f}(\vec{q})q_i$, corresponds to take the covariant derivative

$$-i\frac{\partial}{\partial a_i} + eA_i(\vec{a}) \tag{299}$$

of the inverse Fourier transform of $\hat{f}(\vec{q})$. An interesting remark is that a translation in $\vec{q}$ generated by $\exp(\vec{\alpha} \cdot \vec{D})$, gives rise to a phase transformation on $f(\vec{a})$. First of all, by using the invariance of the integration measure we can check that

$$\hat{f}(\vec{q} + \vec{\alpha}) = e^{\vec{\alpha}\cdot\vec{D}}\hat{f}(\vec{q}) \tag{300}$$

In fact

$$\int_{(\vec{q})} \hat{f}(\vec{q} + \vec{\alpha})x_{\vec{q}}\,(-\vec{a}) = \int_{(\vec{q}-\vec{\alpha})} \hat{f}(\vec{q})x_{\vec{q}-\vec{\alpha}}\,(-\vec{a}) = e^{-i\vec{\alpha}\cdot\vec{a}}f(\vec{a}) \tag{301}$$

Then, we have

$$\int_{(\vec{q})} \left(e^{\vec{\alpha}\cdot\vec{D}}\hat{f}(\vec{q})\right) x_{\vec{q}}\,(-\vec{a}) = \int_{(\vec{q})} \hat{f}(\vec{q}) \left(e^{-\vec{\alpha}\cdot\vec{D}}x_{\vec{q}}(-\vec{a})\right) = e^{-i\vec{\alpha}\cdot\vec{a}}f(\vec{a}) \tag{302}$$

where we have made use of eq. (293). This proves eq. (300), and at the same time our assertion. From eq. (296), this is equivalent to a gauge transformation on the gauge potential $\mathcal{A}_i = \alpha_{ij}a_j$, $\mathcal{A}_i \to \mathcal{A}_i - \partial_i\Lambda$, with $\Lambda = \vec{\alpha}\cdot\vec{a}$. Therefore, we see here explicitly the content of a projective representation in the basis of the functions on the group. One starts assigning the two-form α_{ij}. Given that, one makes a choice for the vector potential. For instance in the previous analysis we have chosen $\alpha_{ij}a_j$. Any possible projective representation corresponds to a different choice of the gauge. In the dual Fourier basis this corresponds to assign a fixed set of operators q_i, with commutation relations determined by a two-form. All the possible projective representations are obtained by translating the operators q_i's. Of course, this is equivalent to say that the projective representations are the central extension of the vector ones, and that they are determined by the cocycles. But the previous analysis shows that the projective representations generate noncommutative spaces, and that the algebraic integration, allowing us to define a Fourier analysis, gives the possibility of establishing calculus rules over these spaces.

Consider now the case $G = Z^D$. Let us introduce an orthonormal basis on the square lattice defined by Z^D, $\vec{e}_{(i)}$, $i = 1, \cdots, D$. Then, any element of the algebra can be reconstructed in terms of a product of the elements

$$U_i = x(\vec{e}_{(i)}) \tag{303}$$

corresponding to a translation along the direction i by one lattice site. In general we will have

$$x(\vec{m}) = e^{i\theta(\vec{m})} U_1^{m_1} \cdots U_D^{m_D}, \qquad \vec{m} = \sum_i m_i \vec{e}_{(i)} \tag{304}$$

with θ a calculable phase factor. The quantities U_i play the same role of $\vec{q}$ of the previous example. The Fourier transform is defined by

$$\hat{f}(\vec{U}) = \sum_{\vec{m} \in Z^D} f(\vec{m}) x_{\vec{U}}(\vec{m}) \tag{305}$$

where the dependence on the representation is expressed in terms of $\vec{U}$, denoting the collections of the U_i's. The inverse Fourier transform is defined by

$$f(\vec{m}) = \int_{\vec{U}} \hat{f}(\vec{U}) x_{\vec{U}}(-\vec{m}) \tag{306}$$

where the integration rule is

$$\int_{(\vec{U})} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},\vec{0}} \tag{307}$$

Therefore, the Fourier transform of U_i is simply $\delta_{\vec{m},\vec{e}_{(i)}}$. The algebraic integration for the vector case is

$$\int_{(\vec{U})} \to \int_0^L \frac{d^D \vec{q}}{L^D} \tag{308}$$

Since the set $\vec{U}$ is within the generators of the algebra, to establish the rules of the calculus is a very simple matter. Eq. (303) is the definition of the set $\vec{U}$, analogous to eq. (283). In place of eq. (290) we get

$$U_i \hat{f}(\vec{U}) U_i^{-1} = e^{-2\alpha_{ij} D_j} \hat{f}(\vec{U}) \tag{309}$$

Here D_j is the j-th component of the derivation $\vec{D}$ which acts upon U_i as

$$D_i U_j = -i\delta_{ij} U_j \tag{310}$$

By choosing $\hat{f}(\vec{U}) = U_k$ we have

$$U_i U_k U_i^{-1} U_k^{-1} = e^{2i\alpha_{ik}} \tag{311}$$

which is the analogue of the commutator among the q_i's. The automorphisms generated by $\vec{D}$ are

$$S(\vec{\phi}) x_{\vec{U}}(\vec{m}) = e^{\vec{\phi}\cdot\vec{D}} x_{\vec{U}}(\vec{m}) = e^{-i\vec{\phi}\cdot\vec{m}} x_{\vec{U}}(\vec{m}) \tag{312}$$

From which we see that

$$U_i \to S(\vec{\phi}) U_i = e^{-i\phi_i} U_i \tag{313}$$

This transformation corresponds to a trivial cocycle. As in the case $G = \mathbb{R}^D$ it gives rise to a phase transformation on the group functions

$$\int_{(\vec{U})} \left(e^{\vec{\alpha}\cdot\vec{D}} \hat{f}(\vec{U}) \right) x_{\vec{U}}(-\vec{m}) = \int_{\vec{U})} \hat{f}(\vec{U}) \left(e^{-\vec{\alpha}\cdot\vec{D}} x_{\vec{U}}(-\vec{m}) \right) = e^{i\vec{\phi}\cdot\vec{m}} f(\vec{m}) \tag{314}$$

Of course, all these relations could be obtained formally, by putting $U_i = \exp(-iq_i)$, with q_i defined as in the case $G = R^D$.

Finally, in the case $G = Z_n^D$, the situation is very much alike Z^D, that is the algebra can be reconstructed in terms of a product of elements

$$U_i = x(\vec{e}_{\,(i)}) \tag{315}$$

satisfying

$$U_i^n = 1 \tag{316}$$

Therefore we will not repeat the previous analysis but we will consider only the case $D = 2$, where U_1 and U_2 can be expressed as[14]

$$(U_1)_{a,b} = \delta_{a,b-1} + \delta_{a,n}\delta_{b,1}, \qquad (U_2)_{a,b} = e^{\frac{2\pi i}{n}(a-1)} \delta_{a,b}, \qquad a, b = 1, \cdots, n \tag{317}$$

The elements of the algebra are reconstructed as

$$x_{\vec{U}}(\vec{m}) = e^{i\frac{\pi}{n} m_1 m_2} U_1^{m_1} U_2^{m_2} \tag{318}$$

The cocycle is now

$$\alpha(\vec{m}_1, \vec{m}_2) = -\frac{2\pi}{n} \epsilon_{ij} m_{1i} m_{2j} \tag{319}$$

In this case we can compare the algebraic integration rule

$$\int_{\vec{U}} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},\vec{0}} \tag{320}$$

with

$$Tr[x_{\vec{U}}(\vec{m})] = n\delta_{\vec{m},\vec{0}} \tag{321}$$

A generic element of the algebra is a $n \times n$ matrix

$$A = \sum_{m_1,m_2=0}^{n-1} c_{m_1 m_2} x_{\vec{U}}(\vec{m}) \tag{322}$$

and therefore

$$\int_{\vec{U}} A = \frac{1}{n} Tr[A] \tag{323}$$

In Section 3 we have shown that the algebraic integration over the algebra of the $n \times n$ matrices $\mathcal{A}_n$ is given by

$$\int_{\mathcal{A}_n} A = Tr[A] \tag{324}$$

implying

$$\int_{\vec{U}} A = \frac{1}{n} \int_{\mathcal{A}_n} A \tag{325}$$

3.9 The example of the algebra on the circle

A particular example of a group algebra is the algebra on the circle defined by

$$z^n z^m = z^{n+m}, \quad -\infty \leq n, m \leq +\infty \tag{326}$$

with z restricted to the unit circle.

$$z^* = z^{-1} \tag{327}$$

This is a group algebra over $\mathbb{Z}$. Defining the ket

$$|z\rangle = \begin{pmatrix} \cdot \\ z^{-i} \\ \cdot \\ 1 \\ z \\ \cdot \\ z^i \\ \cdot \end{pmatrix} \tag{328}$$

the R_i and L_i matrices are given by

$$(R_i)_{jk} = \delta_{i+j,k}, \quad (L_i)_{jk} = \delta_{i+k,j} \tag{329}$$

and from our previous construction, the matrix C is given by

$$(C)_{ij} = (C^{-1})_{ij} = \delta_{i+j,0} \tag{330}$$

or. more explicitly by

$$C = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{331}$$

showing that

$$C: \quad z^i \to z^{-i} \tag{332}$$

In fact

$$(C^{-1}L_iC)_{lp} = \delta_{l,-m}\delta_{i+n,m}\delta_{n,-p} = \delta_{i-p,-l} = \delta_{i+l,p} = (R_i)_{lp} \tag{333}$$

In this case the C matrix is nothing but the complex conjugation ($z \to z^* = z^{-1}$). The completeness relation reads now

$$\int_{(z)} z^i z^{-j} = \delta_{ij} \tag{334}$$

from which

$$\int_{(z)} z^k = \delta_{k0} \tag{335}$$

Our algebraic definition of integral can be interpreted as an integral along a circle C around the origin. In fact we have

$$\int_{(z)} = \frac{1}{2\pi i}\int_C \frac{dz}{z} \tag{336}$$

4 Associative Non Self-Conjugated Algebras: the q-oscillator

A generalization of the bosonic oscillator is the q-bosonic oscillator.[15] We will use the definition given in [16]

$$b\bar{b} - q\bar{b}b = 1 \tag{337}$$

with q real and positive. We assume as elements of the algebra $\mathcal{A}$, the quantities

$$x_i = \frac{z^i}{\sqrt{i_q!}} \tag{338}$$

where z is a complex number,

$$i_q = \frac{q^i - 1}{q - 1} \tag{339}$$

and

$$i_q! = i_q(i-1)_q \cdots 1 \tag{340}$$

The structure constants are

$$f_{ijk} = \delta_{i+j,k}\sqrt{\frac{k_q!}{i_q! j_q!}} \tag{341}$$

and therefore

$$(R_i)_{jk} = \delta_{i+j,k}\sqrt{\frac{k_q!}{i_q! j_q!}}, \qquad (L_i)_{jk} = \delta_{i+k,j}\sqrt{\frac{j_q!}{i_q! k_q!}} \tag{342}$$

In particular

$$(R_1)_{jk} = \delta_{j+1,k}\sqrt{k_q}, \qquad (L_1)_{jk} = \delta_{j-1,k}\sqrt{(k+1)_q} \tag{343}$$

We see that R_1 and L_1 satisfy the q-bosonic algebra

$$R_1 L_1 - q L_1 R_1 = 1 \tag{344}$$

This equation shows that the right- and left-representations are not equivalent for $q \neq -1$ (the Fermi oscillator case). Therefore no C matrix exists and, according to our rules of Section 2.2, we require

$$\int_{(z,z^*)_q} \frac{z^i z^{*j}}{i_q! j_q!} = \delta_{ij} \tag{345}$$

In the case $q = 1$ the integration coincides with the standard integration over complex numbers as used for coherent states [8]

$$\int_{(z,z^*)_1} = \int \frac{dz\, dz^*}{2\pi i} e^{-|z|^2} \tag{346}$$

The integration for the q-oscillator can be expressed in terms of the so called q-integral (see ref.[17]), by using the representation of $n_q!$ as a q-integral

$$n_q! = \int_0^{1/(1-q)} d_q t \; e_{1/q}^{-qt} \; t^n \tag{347}$$

where the q-exponential is defined by

$$e_q^t = \sum_{n=0}^{\infty} \frac{z^n}{n_q!} \tag{348}$$

and the q-integral through

$$\int_0^a d_q t f(t) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n \tag{349}$$

Then the two integrations are related by ($z = |z| \exp(i\phi)$)

$$\int_{(z,z^*)_q} = \int \frac{d\phi}{2\pi} \int d_q(|z|^2) \; e_{1/q}^{-q|z|^2} \tag{350}$$

The Jackson integral is the inverse of the q-derivative

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} \tag{351}$$

In fact, for

$$F(a) = \int_0^a d_q t f(t) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n \tag{352}$$

one has

$$(D_q F)(a) = f(a) \tag{353}$$

as can be checked using

$$F(qa) = qa(1-q) \sum_{n=0}^{\infty} f(aq^{n+1}) q^n = a(1-q) \sum_{n=1}^{\infty} f(aq^n) q^n = F(a) - a(1-q) f(a) \tag{354}$$

The limit $q \to 1$ defines the rules for the integration in the case of the normal bosonic oscillator.

5 Non-Associative Self-Conjugated Algebras: the octonions

We will discuss now how to integrate over the non-associative algebra of octonions (see [18]). This algebra (said also a Cayley algebra) is defined in terms of the multiplication table of its seven imaginary units e_A

$$e_A e_B = -\delta_{AB} + a_{ABC} e_C, \quad A, B, C = 1, \cdots, 7 \tag{355}$$

where a_{ABC} is completely antisymmetric and equal to $+1$ for $(ABC) = (1,2,3)$, (2,4,6), (4,3,5), (3,6,7), (6,5,1), (5,7,2) and (7,1,4). The automorphism group of the algebra is G_2. We define the split basis as

$$\begin{aligned} u_0 &= \frac{1}{2}(1 + ie_7), \qquad u_0^* = \frac{1}{2}(1 - ie_7) \\ u_i &= \frac{1}{2}(e_i + ie_{i+3}), \qquad u_i^* = \frac{1}{2}(e_i - ie_{i+3}) \end{aligned} \tag{356}$$

where $i = 1, 2, 3$. In this basis the multiplication rules are given in Table 2 and automorphism group is $SU(3)$.

Table 2: *Multiplication table for the octonionic algebra.*

	u_0	u_0^*	u_j	u_j^*
u_0	u_0	0	u_j	0
u_0^*	0	u_0^*	0	u_j^*
u_i	0	u_i	$\epsilon_{ijk} u_k^*$	$-\delta_{ij} u_0$
u_i^*	u_i^*	0	$-\delta_{ij} u_0^*$	$\epsilon_{ijk} u_k$

The non-associativity can be checked by taking, for instance,

$$u_i(u_j u_k^*) = u_i(-\delta_{jk} u_0) = 0 \tag{357}$$

and comparing with

$$(u_i u_j)u_k^* = \epsilon_{ijm} u_m^* u_k^* = -\epsilon_{ijk}\epsilon_{kmn} u_n \tag{358}$$

From the ket

$$|u\rangle = \begin{pmatrix} u_0 \\ u_0^* \\ u_i \\ u_i^* \end{pmatrix} \tag{359}$$

one can easily evaluate the matrices R and L corresponding to right and left multiplication. We will not give here the explicit expressions, but one can easily see some properties. For instance, one can evaluate the anticommutator $[R_i, R_j^*]_+$, by using the following relation

$$[R_i, R_j^*]_+|u\rangle = R_i|u\rangle u_j^* + R_j^*|u\rangle u_i = (|u\rangle u_i)u_j^* + (|u\rangle u_j^*)u_i \tag{360}$$

The algebra of the anticommutators of R_i, R_i^* turns out to be the algebra of three Fermi oscillators (apart from the sign)

$$[R_i, R_j^*]_+ = -\delta_{ij}, \quad [R_i, R_j]_+ = 0, \quad [R_i^*, R_j^*]_+ = 0 \tag{361}$$

The matrices R_0 and R_0^* define orthogonal projectors

$$R_0^2 = R_0, \quad (R_0^*)^2 = R_0^*, R_0 R_0^* = R_0^* R_0 = 0 \tag{362}$$

Further properties are

$$R_0 + R_0^* = 1 \tag{363}$$

and

$$R_i^* = -R_i^T \tag{364}$$

Similar properties hold for the left multiplication matrices. This algebra is self-conjugated with the C matrix given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1_3 \\ 0 & 0 & -1_3 & 0 \end{pmatrix} \tag{365}$$

where 1_3 is the 3×3 identity matrix. We have $C^T = C$. The integration rules are easily obtained by looking at the following external product

$$\begin{aligned} C|u\rangle\langle u| &= \begin{pmatrix} u_0 \\ u_0^* \\ -u_i^* \\ -u_i \end{pmatrix} \left(u_0, u_0^*, u_j, u_j^*\right) \\ &= \begin{pmatrix} u_0 & 0 & u_j & 0 \\ 0 & u_0^* & 0 & u_j^* \\ -u_i^* & 0 & \delta_{ij}u_0^* & -\epsilon_{ijk}u_k \\ 0 & -u_i & -\epsilon_{ijk}u_k^* & \delta_{ij}u_0 \end{pmatrix} \end{aligned} \tag{366}$$

Therefore we get

$$\int_{(u)} u_0 = \int_{(u)} u_0^* = 1, \qquad \int_{(u)} u_i = \int_{(u)} u_i^* = 0 \tag{367}$$

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